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EXİSTENCE AND UNİQUENESS OF THE SOLUTİON OF A SECOND ORDER NONLİNEAR DİFFERENTİAL EQUATİON WİTH NONLOCAL CONDİTİONS

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Abstract. In the presented article, the existence and uniqueness of the solution for nonlinear second-order differential equations are investigated. The uniqueness of the solution is based on Banach's fixed-point principle, while the existence is based on Krasnoselskii's fixed point theorem.

Keywords: Second order differential equations, nonlocal boundary conditions, fixed point theory, the existence and uniqueness of the solution.

AMS Subject Classification: 34B10, 34B27, 34K10.

1. Introduction

The mathematical modeling of some real processes leads to the investigation of boundary value problems for nonlinear differential equations. The wide application of boundary value problems in fluid mechanics, mathematical physics, and other scientific fields has made the study of these problems an important branch of science. It is possible to detail [2,5,19,20] applications in this context.

Classical boundary conditions do not take into account some important characteristics of certain processes. This leads to the emergence of nonlocal boundary conditions. Nonlocal conditions establish a relationship between the boundary values of the sought solution and the interior points of the domain.

Boundary value problems for first order differential equations have been studied in [3,12-15,21]. Boundary value problems for second order differential equations have been studied in [1,4,6-9,11,16,17, 18, 22, 23].

2. The formulation of the problem and preliminary results.

In this article, we will investigate the existence and uniqueness of the solution for a second order nonlinear differential equation with nonlocal conditions as follows.

Let us assume that $t \in [0,T]$

$$\begin{cases} x''(t) = f\big(t, x(t)\big) \\ Ax'(0) + Bx'(T) = 0 \\ x(0) = C \end{cases} \qquad (1)$$

a system of differential equations is given. Here, $f:[0,T]\times R^n \to R^n$ is a given continuous function, and $A, B \in R^{n\times n}$, $C \in R^{n\times 1}$ are constant matrices.

Lemma. Let us assume that $f \in C([0,T]\times R^n, R^n)$ and $det(A+B) \neq 0$ Then, the solution of the problem (1) is

$$x(t) = C - (A+B)^{-1}B\int_0^T y(t)dt\,t + \int_0^t (t-\tau)f(\tau, x(\tau))d\tau \qquad (2)$$

Proof. $x''(t) = f\big(t, x(t)\big)$ let us integrate the equality from 0 to t . As a result,

$$x'(t) = x'(0) + \int_0^t f(\tau, x(\tau))d\tau \qquad (3)$$

we obtain , where $x'(0)$ is an unknown n dimensional constant vector.

$Ax'(0) + Bx'(T) = 0$

using the condition, we can determine the vector $x'(0)$. It is clear that

$$Ax'(0) + B(x'(0) + \int_0^T f\big(t, x(t)\big)dt) = 0$$

From here,

$$x'(0) = -(A+B)^{-1}B\int_0^T f\big(t, x(t)\big)\,dt. \qquad (4)$$

Now, by considering the equality (5) in (4), then

$$x'(t) = -(A+B)^{-1}B\int_0^T f\big(t, x(t)\big)\,dt + \int_0^t f(\tau, x(\tau))d\tau \qquad (5)$$

The final equation let's integrate equation (5) again from 0 to t

$x(t) = x(0) - (A+B)^{-1}B\int_0^T f\big(t, x(t)\big)\,dt\,t + \int_0^t (t-\tau)f(\tau, x(\tau))d\tau$

We will obtain the equation. If we consider the condition $x(0) = c$, we obtain equation (2). The lemma is proved.

Let's introduce the $P: R^n \to R^n$ operator as follows. Here

$$(Px)(t) = C - (A+B)^{-1}B\int_0^T f\big(t, x(t)\big)dt\,t + \int_0^t (t-\tau)f(\tau, x(\tau))d\tau \qquad (6)$$

If we consider the lemma we proved above, problem (1)

$$x = Px \qquad (7)$$

is equivalent to the operator equation.

As it can be seen, the fixed point of operator (6) is a solution to the operator equation (7) or problem (1).

3. The existence of the solution.

In this section, we will find a sufficient condition for the existence of a solution to problem (1). For this, we will use Krasnoselskii's fixed point theorem.

Theorem 1. (Krasnoselskii's fixed point theorem) [10]. Let Y be a closed bounded, convex, and nonempty subset of a Banach space X. Let P_1, P_2 be the operators mapping Y into X such that

(i) $P_1y_1 + P_2y_2 \in Y$ whenever $y_1, y_2 \in Y$;

(ii) P_1 is compact and continuous ;

(iii) P_2 is a contraction mapping.

Then there exists $y \in Y$ such that $y = P_1y + P_2y$.

Theorem 2. Let us assume that $f: [0,T] \times R^n \to R^n$ the function is continuous and satisfies the following conditions:

(A1) $|f(t,x) - f(t,y)| \le l|x-y|, \quad \forall t \in [0,T], \quad l > 0, \; x,y \in R^n$;

(A2) There exists a function $\|\mu\| \in C([0,T], R^+)$ such that $\|\mu\| = \max_{[0,T]} |\mu(t)|$,

$$|f(t,x)| \le \|\mu\|, \forall (t,x,x') \in [0,T] \times R^n.$$

If

$$lST < 1 \tag{8}$$

holds, then there exists at least one solution to the boundary problem (1) on the interval $[0,T]$.

Proof. $B_r = \{x \in P: \; \|x\| \le r\}$ let's consider the closed sphere. Here, $r \ge \|\mu\|Q + |c|$

$$Q = \frac{1}{2}T^2 + ST \tag{9}$$

Let's define the operators P_1 and P_2 defined on the sphere B_r

$$(P_1x)(t) = \int_0^t (t-s)\, f(s, x(s))ds,$$

$$(P_2x)(t) = C - (A+B)^{-1}B\int_0^T f(t, x(t))dt\,t.$$

Show that $P = P_1 + P_2$

For any $x, y \in B_r$

$$\|P_1x + P_2y\| = \left| C + \int_0^t (t-s)f(s,x(s))ds - (A+B)^{-1}B\int_0^T f(t,y(t))dt\,t \right|$$

$$\le |C| + \|\mu\| \left[\frac{T^2}{2} + \left|(A+B)^{-1}B\right|T\right] = |C| + \|\mu\| \left(\frac{T^2}{2} + ST\right)$$

$$\le |C| + \|\mu\|Q \le r$$

Since $P_1x + P_2y \in B_r$ holds, the condition (i) of Krasnoselskii's theorem is satisfied. Using the condition (A1)

$$\|P_2x - P_2y\| =$$

$$= \left| \int_0^T (A+B)^{-1}Bf(s,x(s))ds - \int_0^T -(A+B)^{-1}Bf(s,y(s))ds \right|$$

$$\le \int_0^T S \cdot |f(s,x(s) - f(s,y(s))|ds \le lST\|x-y\|$$

Since condition (8) is satisfied, it follows that the P_2 operator is compact.

Now, let us show that the P_1 operator is completely continuous, meaning it is compact and continuous. Note that the continuity of the function f implies that the P_1 operator is continuous. It is also clear that the P_1 operator is uniformly bounded on B_r and

$$\|P_1x\| \le \|\mu\|\frac{T^2}{2}$$

Let us assume that $max|f(t,x)| = \bar{f}$ and $0 < t_1 < t_2 < T$ Then

$$|(P_1x)(t_2)-(P_1x)(t_1)| = \left|\int_0^{t_2}(t_2-s)f(s,x(s))ds - \int_0^{t_1}(t_1-s)f(s,x(s))ds\right|$$
$$= \left|\int_0^{t_1}((t_2-s)-(t_1-s))f(s,x(s))ds + \int_{t_1}^{t_2}(t_2-s)f(s,x(s))ds\right|$$
$$\leq \overline{f}\left|(t_2-t_1)t_1 + \frac{(t_2-t_1)^2}{2}\right| \to 0 \quad t_2 \to t_1$$

The last relation is independent of $x \in B_r$. Thus, all conditions of Krasnoselskii's theorem are satisfied.

Therefore, there exists at least one solution to the nonlocal boundary problem (1).

4. The uniqueness of the solution.

In this section, we will find sufficient conditions that ensure the uniqueness of the solution to the nonlocal boundary problem (1) with the help of Banach's fixed-point principle [10].

Theorem 3. Let us assume that $f: [0,T] \times R^n \to R^n$ is a continuous function and the condition (A1) is satisfied. If

$$lQ < 1$$

holds, The boundary problem (1) has a unique solution on the interval $[0,T]$, here, Q is defined by the expression (9).

Proof. $\max_{[0,T]} |f(t,0)| = M$ let us note. $r \geq \frac{|C|+MQ}{1-lQ}$ let us note and show that $PB_r \subset B_r$, here B_r=$\{x \in P: \|x\| \leq$ r $\}$. Any $x \in B_r$ and $t \in [0,T]$

$$\left|f(t,x(t))\right| = \left|f(t,x(t)) - f(t,0) + f(t,0)\right| \leq$$
$$\leq |f(t,x(t)) - f(t,0)| + f(t,0) \leq l\|x\| + M \leq lr + M$$

the inequality is true. Then, for $x \in B_r$

$$\|(Px)\| = max\left|C + \int_0^T -(A+B)^{-1}Bf(s,x(s))dt + \int_0^t (t-s)f(s,x(s))ds\right|$$
$$\leq |C| + (lr+M)max\left\{\int_0^T |(A+B)^{-1}B|ds + \frac{T^2}{2}\right\} \leq |C| + (lr+M)Q \leq r$$

Here, Q is defined by the expression (10). This relation shows that $PB_r \subset B_r$,

Now, let us show that the operator P is a contraction. Any for $x, y \in P$

$$\|(Px)-(Py)\| =$$
$$= max\left|-(A+B)^{-1}B\int_0^T f(t,x(t))dt\,t + \int_0^t (t-s)f(s,x(s))ds\right.$$

$$+\int_0^T (A+B)^{-1}Bf(s,y(s))ds-\int_0^t (t-s)f(s,y(s))ds\Bigg|$$

$$\leq max\int_0^T |(A+B)^{-1}B|\ |f(s,x(s))-f(s,y(s))|ds$$

$$+max\int_0^t |t-s|\cdot|f(s,x(s))-f(s,y(s))|ds$$

$$\leq Sl\int_0^T |x(s)-y(s)|ds+l\int_0^T (t-s)|x(s)-y(s)|ds$$

$$\leq\left(SlT+l\frac{T^2}{2}\right)\|x-y\|=Ql\|x-y\|.$$

It follows from condition $Ql < 1$ that the operator P is a contraction. Thus, all the conditions of Banach's fixed-point theorem are satisfied, and as a result, it follows that the nonlocal boundary problem (1) has a unique solution on the interval $[0,T]$.

Now, investigate the continuous dependence of the solution of problem (1) on the right-hand side of the boundary condition.

Theorem 4. Assume that condition (A1) holds and $lQ < 1$. Then, for any $C_1, C_2 \in R^n$ and the corresponding

$$\begin{cases} x_i''(t)=f(t,x_i(t)) \\ Ax_i(0)+Bx_i(T)=0 \\ x_i(0)=C_i,\ \ i=1,2, \end{cases} \tag{10}$$

for the solutions $x_1(t)$ and $x_2(t)$ of the border issue

$$\|x_1(t)-x_2(t)\|\leq(1-lQ)^{-1}\|C_1-C_2\|$$

is correct.

Proof. Assume that $C_1, C_2 \in R^n$ is arbitrary points, $x_1(t)$ and $x_2(t)$ are solutions to the corresponding boundary problem. Then

$$\begin{aligned} x_1(t)-x_2(t)&=C_1-C_2\\ &-(A+B)^{-1}B\int_0^T f(t,x_1(t))dt\,t+\int_0^t(t-\tau)f(\tau,x_1(\tau))d\tau\\ &+(A+B)^{-1}B\int_0^T f(t,x_2(t))dt-\int_0^t(t-\tau)f(\tau,x_2(\tau))d\tau \end{aligned} \tag{11}$$

From this equality

$$|x_1(t)-x_2(t)|\leq|C_1-C_2|+TSl\|x_1-x_2\|+l\frac{T^2}{2}\|x_1+x_2\|$$
$$=|C_1-C_2|+Ql\|x_1-x_2\|.$$

from here,

$$\|x_1-x_2\|\leq\frac{\|C_1-C_2\|}{1-Ql}.$$

The theorem is proved.

5. Conclusion.

In the paper, a boundary value problem with a nonlocal condition involving a small point has been studied. By imposing certain conditions on the initial data and applying Krasnoselskii's fixed point theorem, sufficient conditions for the existence of at least one solution to the boundary value problem have been established. Using Banach's contraction mapping principle, the existence of a unique solution to the considered boundary value problem has been proven.

The scheme used in the paper can be applied to more general boundary value problems. For example, consider the following nonlocal boundary value problem.

$$x'(t) = f\big(t, x(t), x'(t)\big), \qquad t \in [0,T]$$
$$A_1 x'(0) + B_1 x'(T) = C_1$$
$$A_2 x(0) + B_2 x(T) = C_2$$

Here, $A_i, B_i \in R^{n\times n}$, $C_i \in R^n$ $(i = 1,2)$, $f: [0,T] \times R^n \times R^n \to R^n$ is a continuous function.

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